

c-map as $c = 1$ string

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ABSTRACT: We show the existence of a duality between the c-map space describing the universal hypermultiplet at tree level and the matrix model description of two-dimensional string theory compactified at a self-dual radius and perturbed by a sine-Liouville potential. It appears as a particular case of a general relation between the twistor description of four-dimensional quaternionic geometries and the Lax formalism for Toda hierarchy. Furthermore, we give an evidence that the instanton corrections to the c-map metric coming from NS5-branes can be encoded into the Baker-Akhiezer function of the integrable hierarchy.

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1. Introduction

Integrability is a very powerful tool to find exact solutions and it is always a great achievement when a theory is proven to possess an integrable structure. A particularly interesting class of theories where various appearances of integrability have been observed during the last few years is characterized by the presence of $N = 2$ supersymmetry. Remarkably, it appears both in the gauge [1, 2, 3, 4, 5] and in the string theory context [6, 7]. However, an exhaustive explanation of these observations is missing so far.

In this note we reveal one more instance of an integrable structure in theories with $N = 2$ supersymmetry. Namely, we observe that the four-dimensional quaternion-Kähler (QK) space obtained by the local c-map construction [8, 9] is in a certain sense dual to a sine-Liouville perturbation of the $c = 1$ string theory compactified at the self-dual radius. The former represents the tree level moduli space of the so called universal hypermultiplet [10] arising in compactifications of type II strings on a rigid Calabi-Yau. On the other hand, the latter is known to be an exactly solvable theory which is described by Toda integrable hierarchy [11, 12]. In particular, the lowest equation of this hierarchy is given by the Toda equation, which is also known to parametrize four-dimensional quaternionic metrics with one isometry

[13]. The duality which we report here is based on the observation that the two systems are described by the same solution of the Toda equation.

However, the duality goes much further as it allows to identify also various quantities characterizing either the c -map or the $c = 1$ string. These identifications originate in a general correspondence between a twistor description of quaternionic manifolds and the Lax formalism for integrable hierarchies [14]. The duality between the c -map and the $c = 1$ string is just its simplest example.

The above mentioned correspondence is not at all surprising. First of all, the existence of integrable hierarchies governing the four-dimensional hyperkähler (HK) geometry is very well known (see, for instance, [15, 16, 17]). Moreover, recently it was realized that hyperkähler manifolds with a rotational isometry endowed with a hyperholomorphic line bundle can be associated with QK manifolds of the same dimension also possessing an isometry [18, 19]. In particular, this QK/HK correspondence identifies the rigid and local c -map spaces and makes a contact between QK geometry and integrability. Although, this might seem to imply that the duality we propose is rather trivial from the mathematical point of view, its physical realization is quite fascinating as it suggests that the hypermultiplet sector of $N = 2$ superstring compactifications and non-critical $c = 1$ string theory are closely related to each other. We are not aware of any examples where some perturbation of two-dimensional string theory was related to the hypermultiplet moduli space.

Of course, this duality would become really interesting if it can be extended beyond the classical limit and used to extract quantum corrections on one side from the known results on the other side. We leave the systematic study of this problem for the future. Nevertheless, in this note we make the first step in this direction. Namely, we observe that the holomorphic functions on the twistor space of the c -map, which have been shown to generate NS5-brane instanton corrections to the tree level universal hypermultiplet, can be interpreted in terms of the Baker-Akhiezer wave function of the Toda hierarchy. Together with previous findings [3, 7], this suggests that the whole non-perturbative picture describing $N = 2$ string compactifications possesses an integrable structure where the relations discussed here will find their natural place.

2. Toda lattice hierarchy

We start from a brief review of two-dimensional Toda lattice hierarchy [20] which provides the general framework for what we are going to discuss. This integrable hierarchy describes many physical phenomena which, in particular, include a certain class of deformations of $c = 1$ string theory. More precisely, it encompasses all perturbations generated by the spectrum of tachyon operators corresponding to the $c = 1$ string compactified on a circle [11, 12]. Furthermore, the Lax formalism of this hierarchy turns out to be closely related to the matrix model description of the $c = 1$ string, which is provided by Matrix Quantum Mechanics (MQM). Therefore, first we present this formalism, then we explain its interpretation from the MQM point of view, and finally we give a particular solution of the Toda hierarchy which describes $c = 1$ string theory with the so-called sine-Liouville perturbation. Although we do not need the Toda hierarchy in its full generality to explain the relation with the c -map spaces, it is nevertheless useful to recall it as we want to discuss further extensions and generalizations of the proposed duality.

2.1 Lax formalism

Let us first give a formal definition of the Toda lattice hierarchy. To this end, introduce two semi-infinite series

$$L = r(s)\hat{\omega} + \sum_{k=0}^{\infty} u_k(s)\hat{\omega}^{-k}, \quad \bar{L} = \hat{\omega}^{-1}r(s) + \sum_{k=0}^{\infty} \hat{\omega}^k \bar{u}_k(s), \quad (2.1)$$

where s is a discrete variable labeling the nodes of an infinite lattice, $\hat{\omega} = e^{\hbar\partial/\partial s}$ is the shift operator in s , and the Planck constant \hbar plays the role of a spacing parameter. The operators (2.1) are called Lax operators. Their coefficients r , u_k and \bar{u}_k are taken to be functions of two infinite sets of “times” $\{t_{\pm k}\}_{k=1}^{\infty}$. To each time variable one associates a Hamiltonian $H_{\pm k}$ generating an evolution along $t_{\pm k}$ by the usual rule

$$\begin{aligned} \hbar \frac{\partial L}{\partial t_k} &= [H_k, L], & \hbar \frac{\partial \bar{L}}{\partial t_k} &= [H_k, \bar{L}], \\ \hbar \frac{\partial L}{\partial t_{-k}} &= [H_{-k}, L], & \hbar \frac{\partial \bar{L}}{\partial t_{-k}} &= [H_{-k}, \bar{L}]. \end{aligned} \quad (2.2)$$

This system represents the Toda hierarchy if the Hamiltonians satisfy an additional requirement, namely that they can be expressed through the Lax operators (2.1) as follows

$$H_k = (L^k)_{>} + \frac{1}{2}(L^k)_0, \quad H_{-k} = (\bar{L}^k)_{<} + \frac{1}{2}(\bar{L}^k)_0, \quad (2.3)$$

where the symbol $(\)_{>}$ means the positive (negative) part of the series in the shift operator $\hat{\omega}$ and $(\)_0$ denotes the constant part. The integrability follows from the existence of an infinite set of commuting flows generated by $\hbar \frac{\partial}{\partial t_k} - H_k$ whose commutativity is a consequence of (2.2).

This definition shows that the Toda hierarchy is a collection of non-linear equations of the finite-difference type in s and differential with respect to t_k for the coefficients $r(s, t)$, $u_k(s, t)$ and $\bar{u}_k(s, t)$. Its hierarchic structure is reflected in the fact that one obtains a closed equation on the first coefficient $r(s, t)$ and its solution provides the input information for the following equations. It is easy to show that $r(s, t)$ should satisfy a discrete version of the Toda equation

$$\hbar^2 \frac{\partial^2 \log r^2(s)}{\partial t_1 \partial t_{-1}} = 2r^2(s) - r^2(s + \hbar) - r^2(s - \hbar), \quad (2.4)$$

which gives the name to the hierarchy.

For our purposes it is important to introduce the following Orlov-Shulman operators [21]

$$M = \sum_{k \geq 1} k t_k L^k + s + \sum_{k \geq 1} v_k L^{-k}, \quad \bar{M} = - \sum_{k \geq 1} k t_{-k} \bar{L}^k + s - \sum_{k \geq 1} v_{-k} \bar{L}^{-k}. \quad (2.5)$$

The coefficients $v_{\pm k}(s, t)$ are supposed to be found from the condition on the commutators of M and \bar{M} with the Lax operators

$$[L, M] = \hbar L, \quad [\bar{L}, \bar{M}] = -\hbar \bar{L}. \quad (2.6)$$

The importance of the Orlov-Shulman operators comes from the fact that they can be considered as perturbations of the simple operators of multiplication by the discrete variable s , whereas the Lax operators appear to be a “dressed” version of the shift operator $\hat{\omega}$. Indeed, if one requires that $v_{\pm k}$ vanish when all $t_{\pm k} = 0$, then in this limit $M = \bar{M} = s$, $L = \hat{\omega}$ and

$\bar{L} = \hat{\omega}^{-1}$. Moreover, the commutation relations (2.6) can be recognized as a dressed version of the trivial relation

$$[\hat{\omega}, s] = \hbar \hat{\omega}. \quad (2.7)$$

Another application of the Orlov–Shulman operators is that they lead directly to the notion of τ -function. Indeed, one can show that their coefficients must satisfy [22]

$$\frac{\partial v_k}{\partial t_l} = \frac{\partial v_l}{\partial t_k}. \quad (2.8)$$

This implies that there exists a generating function $\tau_s[t]$ of $v_{\pm k}$ such that

$$v_k(s, t) = \hbar^2 \frac{\partial \log \tau_s[t]}{\partial t_k}. \quad (2.9)$$

It is called the τ -function of Toda hierarchy. It encodes all information about a particular solution. For example, it allows to extract the first coefficient in the expansion of the Lax operators

$$r^2(s - \hbar) = \frac{\tau_{s+\hbar} \tau_{s-\hbar}}{\tau_s^2}. \quad (2.10)$$

Plugging this relation into (2.4), one obtains that the τ -function is subject to the Toda equation. In physical applications the τ -function usually coincides with partition functions and the coefficients v_k are the one-point correlators of the operators generating the commuting flows H_k .

Finally, note that the Toda hierarchy can be equivalently represented as an eigenvalue problem given by the following equations

$$x\Psi = L\Psi, \quad \hbar x \frac{\partial \Psi}{\partial x} = M\Psi, \quad \hbar \frac{\partial \Psi}{\partial t_k} = H_k\Psi, \quad (2.11)$$

plus similar equations for $\bar{\Psi}$ where L, M should be replaced by $\bar{L}, -\bar{M}$. The eigenfunction $\Psi(x; s, t)$ is known as the *Baker-Akhiezer function*. Often it has a direct physical interpretation and plays a very important role.

2.2 String equations

The Toda hierarchy provides a universal description for many integrable systems because the equations of the hierarchy have many different solutions. There are two ways to select a particular solution. The most obvious one is to provide an initial condition which can be, for instance, the τ -function for vanishing times corresponding physically to the partition function of a non-perturbed system. However, the Toda equations involve partial differential equations of high orders and require to know not only the τ -function at $t_{\pm k} = 0$ but also its derivatives. Therefore, it is not always clear whether the non-perturbed function is sufficient to restore the full τ -function.

Fortunately, there is another way to select a unique solution of the Toda hierarchy. It relies on the use of the so-called *string equations* which are some algebraic relations between the Lax and Orlov–Shulman operators. The advantage of the string equations is that they represent, in a sense, already a partially integrated version of the hierarchy equations. For example, instead of solving a second order differential equation on the τ -function, they reduce the problem to certain algebraic and first order differential equations.

It is important, however, that the string equations cannot be chosen arbitrary because they should be consistent with the commutation relations (2.6). For example, if they are given by two equations of the following type

$$\bar{L} = f(L, M), \quad \bar{M} = g(L, M), \quad (2.12)$$

the operators defined by the functions f and g must satisfy

$$[f(\hat{\omega}, s), g(\hat{\omega}, s)] = -\hbar f. \quad (2.13)$$

2.3 Dispersionless limit

In this paper we are mostly interested only in the classical limit of the Toda hierarchy, which is obtained by taking the spacing parameter \hbar to zero. In this limit one recovers what is known as *dispersionless Toda hierarchy* [22]. It is described by the same equations as above with the only difference that all commutators should be replaced by Poisson brackets and all operators are considered now as functions on the phase space parametrized by s and ω with the symplectic structure induced from (2.7)

$$\{\omega, s\} = \omega. \quad (2.14)$$

As in the full theory, a solution of the dispersionless Toda hierarchy is completely characterized by a dispersionless τ -function. In fact, it is better to talk about free energy since it is the logarithm of the full τ -function that can be represented as a series in \hbar

$$\log \tau = \sum_{n \geq 0} \hbar^{-2+2n} F_n. \quad (2.15)$$

The dispersionless limit is described by the first coefficient of this series, the dispersionless free energy F_0 . It satisfies the classical limit of the Toda equation

$$\partial_{t_1} \partial_{t_{-1}} F_0 + e^{\partial_s^2 F_0} = 0, \quad (2.16)$$

and can be selected from all solutions by the same string equations (2.12) as in the quantum case.

The quasiclassical asymptotics of the Baker-Akhiezer function is given by the usual WKB approximation. Since Ψ diagonalizes the Lax operator, it can be considered as a wave function representing the quantum state of the system in the L -representation. The conjugate variable is provided by M which implies that

$$\Psi \sim \exp \left\{ \frac{1}{\hbar} \int^{\log x} M d \log L \right\}, \quad (2.17)$$

where M is considered as a function of L , s and t_k . Using (2.5), one finds

$$\Psi = e^{\frac{1}{\hbar} S + O(\hbar^0)}, \quad S = \sum_{k \geq 1} t_k x^k + s \log x + \frac{1}{2} \phi - \sum_{k \geq 1} \frac{v_n}{n} x^{-n}. \quad (2.18)$$

The integration constant ϕ can be related to the free energy as $\phi(s, t) = -\partial_s F_0$.

2.4 Example: MQM with sine-Liouville perturbation

Two-dimensional or non-critical $c = 1$ string theory is known to be an example of integrable model described by the Toda hierarchy. However, this is difficult to see in its CFT formulation. On the other hand, the integrability becomes transparent in its matrix model description in terms of Matrix Quantum Mechanics (see [23, 24] for a review).

To see how it appears, one should restrict to the singlet sector of MQM where all angular degrees of freedom are integrated out. The remaining matrix eigenvalues describe a system of free one-dimensional fermions in the inverse oscillator potential. The target space description of string theory arises as an effective theory of collective excitations of the free fermions. In particular, different backgrounds correspond to different states of the Fermi sea: the simplest linear dilaton background is dual to the ground state, whereas various perturbed backgrounds with a non-trivial condensate of the tachyon field come from certain time-dependent states.

The most convenient way to describe the dynamics of these free fermions is to use the light-cone coordinates [12] $x_{\pm} = \frac{x \pm p}{\sqrt{2}}$ where (x, p) are the coordinates on the phase space of one fermion. It is clear that x_{\pm} satisfy the canonical commutation relations

$$\{x_+, x_-\} = 1. \quad (2.19)$$

In the quasiclassical approximation the problem reduces to finding the exact form of the profile of the Fermi sea in the phase space, given its asymptotic form at $x_{\pm} \rightarrow \infty$. Here we are interested in deformations generated by the spectrum of the theory compactified on a circle of radius R . Then the profile of the Fermi sea is determined by the compatibility of the following two equations

$$x_+ x_- = M_{\pm}(x_{\pm}) = \pm \frac{1}{R} \sum_{k \geq 1} k t_{\pm k} x_{\pm}^{k/R} + \mu \pm \frac{1}{R} \sum_{k \geq 1} v_{\pm k} x_{\pm}^{-k/R}. \quad (2.20)$$

Here $x_{\pm}^{k/R}$ correspond to the tachyon vertex operators with momentum k/R , $t_{\pm k}$ are their coupling constants, μ is the Fermi level, and $v_{\pm k}$ are to be found as functions of $t_{\pm k}$ and μ . For vanishing coupling constants, the equations (2.20) describe the hyperbola $x_+ x_- = \mu$ which is the trajectory of a free fermion in the inverse oscillator potential.

Comparing the r.h.s. of (2.20) with (2.5), one immediately sees that M_{\pm} are related to the Orlov-Shulman operators. In fact, one can establish a precise correspondence between all data of the Lax formalism and MQM quantities:

- the lattice parameter s is identified with the Fermi level μ ;
- the Lax operators coincide with the light-cone coordinates, $L = x_+^{1/R}$, $\bar{L} = x_-^{1/R}$;
- the Orlov-Shulman operators encode the asymptotics of the Fermi sea and are given by $M = R M_+$, $\bar{M} = R M_-$;
- the Baker-Akhiezer function coincides with the perturbed one-fermion wave function;
- the string equations are the same as the equations for the profile of the Fermi sea and correspond to the choice $f = (M/R)^{1/R} L^{-1}$, $g = M$.

The commutation relation (2.19) then follows from a combination of (2.6) with the string equations. Finally, the classical limit of the shift operator ω turns out to be a uniformization parameter for the Riemann surface described by (2.20).¹

Note that the two Baker-Akhiezer functions, Ψ and $\bar{\Psi}$, give one-fermion wave functions in the two chiral representations, x_+ and x_- , respectively. The requirement that they represent the same quantum state is precisely what generates the profile equations (2.20). Due to this condition, the two wave functions are related by a Fourier transform.

Restricting to a finite set of non-vanishing couplings $t_{\pm k}$, it is straightforward to find an exact solution, i.e. an explicit form of the Lax and Orlov-Shulman operators as well as the τ -function of this system. We are interested in the particular case $R = 1$ and $t_{\pm k} = 0$ for all $k \geq 2$. The perturbation induced by the couplings $t_{\pm 1}$ is known to correspond to the sine-Liouville operator on the string worldsheet, and the compactification radius $R = 1$ is the self-dual point with respect to T-duality which leads to an enhanced symmetry. The corresponding solution of the Toda hierarchy is given by [12]

$$F_0 = \frac{1}{2} \mu^2 \left(\log \mu - \frac{3}{2} \right) - \mu t_1 t_{-1}, \quad x_{\pm} = \sqrt{\mu} \omega^{\pm 1} \mp t_{\mp 1}. \quad (2.21)$$

3. Four-dimensional QK spaces and integrability

Now we turn to a seemingly completely different subject — quaternionic geometry, which is characterized by the existence of three (almost) complex structures J^i satisfying the algebra of quaternions and appears in two forms, hyperkähler and quaternion-Kähler. In fact, it is well known that in four-dimensions HK spaces possess an integrable structure, which in the presence of an isometry is given by the dispersionless Toda hierarchy presented in the previous section. Moreover, as we explained in the introduction, the QK/HK correspondence allows to transfer these results to the realm of four-dimensional QK geometry. In this section we establish precise relations between the quantities used to describe QK spaces with one Killing vector and the ones entering the Lax formalism.

3.1 Toda ansatz

The most direct way to see the appearance of Toda in the description of QK spaces is to recall that in four dimensions QK manifolds coincide with self-dual Einstein spaces with a non-vanishing cosmological constant. Such spaces are known to be classified by solutions of a single non-linear differential equation [25]. In the presence of an isometry, it can be reduced to the three-dimensional continuous Toda equation

$$\partial_z \partial_{\bar{z}} T + \partial_{\rho}^2 e^T = 0, \quad (3.1)$$

which is identical to the equation (2.16) on the dispersionless free energy of Toda hierarchy. In terms of the solution $T(\rho, z, \bar{z})$, the metric is given by [26]

$$ds_{\mathbb{Q}}^2 = -\frac{3}{\Lambda} \left[\frac{P}{\rho^2} (d\rho^2 + 4e^T dz d\bar{z}) + \frac{1}{P\rho^2} (d\theta + \Theta)^2 \right], \quad (3.2)$$

¹In addition, to describe the theory in Lorentzian signature, the Planck constant should be continued analytically to imaginary values, $\hbar_{\text{Toda}} = i\hbar_{\text{MQM}}$.

where the isometry acts as a shift in the coordinate θ . Here $P \equiv 1 - \frac{1}{2} \rho \partial_\rho T$, Λ is the cosmological constant, and Θ is a one-form such that

$$d\Theta = i(\partial_z P dz - \partial_{\bar{z}} P d\bar{z}) \wedge d\rho - 2i \partial_\rho (P e^T) dz \wedge d\bar{z}. \quad (3.3)$$

The integrability condition for (3.3) follows from (3.1). Of course, not only the metric, but also all geometric data such as quaternionic two forms, Levi-Civita connection, etc., can be expressed through T . We refer to [27] for their explicit expressions.

Comparing the Toda equations (3.1) and (2.16), one immediately finds the following identifications

$$z, \bar{z} \leftrightarrow t_{\pm 1}, \quad \rho \leftrightarrow s, \quad T \leftrightarrow \partial_s^2 F_0. \quad (3.4)$$

In principle, one can go further and find similar relations for other quantities. However, as we already know, working with differential equations is not the best way to describe a solution of the hierarchy. It is much better to use string equations which turn out to have a certain twistorial interpretation [15, 14]. Similarly, QK manifolds have a much more powerful description in terms of their twistor spaces [28]. As we will see now, the two become essentially identical.

3.2 Twistor space and Lax formalism

The twistor space \mathcal{Z} of a QK manifold \mathcal{M} is a \mathbb{CP}^1 bundle over \mathcal{M} , whose connection is given by the $SU(2)$ part \vec{p} of the Levi-Civita connection² on \mathcal{M} and the \mathbb{CP}^1 fiber corresponds to the sphere of almost complex structures J^i . The twistor space carries a so called *complex contact structure* [29, 30], which is represented by a holomorphic one-form \mathcal{X} such that the associated top-form $\mathcal{X} \wedge (d\mathcal{X})^n$ is nowhere vanishing. Locally it can be written as

$$\mathcal{X}^{[i]} = \frac{2}{t} e^{\Phi^{[i]}} (dt + p^+ - ip^3 t + p^- t^2), \quad (3.5)$$

where the index $[i]$ labels open patches of a covering of \mathcal{Z} , t parametrizes the \mathbb{CP}^1 fiber, and the function $\Phi^{[i]}(x^\mu, t)$ is known as the contact potential. In the case we are interested in, when \mathcal{M} features one Killing vector, the contact potential is actually real, globally well defined, and t -independent so that one can put $\Phi^{[i]} = \phi(x^\mu)$.

Locally, it is always possible to choose such coordinates that the contact form \mathcal{X} acquires a standard form. Restricting to the case $\dim_{\mathbb{R}} \mathcal{M} = 4$, this means that in a patch $\mathcal{U}_i \subset \mathcal{M}$ one can write

$$\mathcal{X}^{[i]} = d\alpha^{[i]} + \xi^{[i]} d\tilde{\xi}^{[i]}, \quad (3.6)$$

where $(\xi, \tilde{\xi}, \alpha)$ are analogous to holomorphic Darboux coordinates of complex symplectic geometry. Generically, these coordinates must be regular in \mathcal{U}_i . The only exception is the coordinates defined around the north ($\{t = 0\} \in \mathcal{U}_+$) and the south ($\{t = \infty\} \in \mathcal{U}_-$) poles of \mathbb{CP}^1 . In these patches the Darboux coordinates are assumed to have the following expansions

$$\begin{aligned} \xi^{[\pm]}(t) &= \mathcal{R} t^{\mp 1} + \sum_{n=0}^{\infty} \xi_n^{[\pm]} t^{\pm n}, & \tilde{\xi}^{[\pm]}(t) &= \sum_{n=0}^{\infty} \tilde{\xi}_n^{[\pm]} t^{\pm n}, \\ \alpha^{[\pm]}(t) &= \sum_{n=0}^{\infty} \alpha_n^{[\pm]} t^{\pm n} - 2c \log t, \end{aligned} \quad (3.7)$$

²Recall that the holonomy group of a $4n$ -dimensional QK manifold is contained in $Sp(n) \times SU(2)$.

where \mathcal{R} is a real function on \mathcal{M} and c is a real number known as *anomalous dimension*. Thus, $\xi^{[\pm]}$ have a simple pole and $\alpha^{[\pm]}$ have a logarithmic singularity controlled by c . The latter has an important physical meaning generating the one-loop correction to the local c-map [28].

On the overlap of two patches $\mathcal{U}_i \cap \mathcal{U}_j$, the two coordinate systems are related by a complex contact transformation. Such transformations are generated by holomorphic functions $H^{[ij]}(\xi^{[i]}, \tilde{\xi}^{[j]}, \alpha^{[j]})$, which we call transition functions. Together with the anomalous dimension, they encode all geometric information about \mathcal{M} and its twistor space \mathcal{Z} . To extract the metric from them, the most non-trivial step is to find the Darboux coordinates as functions of t and coordinates on \mathcal{M} . Although this cannot be accomplished generically, it is possible to write certain integral equations on the Darboux coordinates in terms of $H^{[ij]}$. We refrain from writing them explicitly and refer to [31, 32] for more details.

This twistor description has been related to the one based on the Toda ansatz (3.2) in [27]. For our purposes it is sufficient to provide the relation between the coordinates, contact and Toda potentials, which read as follows

$$\rho = e^\phi, \quad z = \frac{1}{2} \tilde{\xi}_0^{[+]}, \quad \theta = \frac{1}{2} \text{Im } \alpha_0^{[+]}, \quad T = 2 \log \mathcal{R}. \quad (3.8)$$

Note that the only effect of the anomalous dimension is to shift the contact potential introduced in (3.5), i.e. $e^\phi = e^\phi|_{c=0} - c$. The identification (3.8) nicely maps this shift to the ambiguity in the solution of the Toda equation, $\tilde{T}(\rho, z, \bar{z}) = T(\rho + c, z, \bar{z})$. Although the two solutions, T and \tilde{T} , are related by a coordinate change, due to the explicit dependence of the metric (3.2) on ρ , the geometry of the corresponding QK manifold is strongly affected by the parameter c .

After this very brief review of the twistor description of QK spaces, one can easily see how it is embedded into the Lax formalism of the dispersionless Toda hierarchy. Indeed, comparing the expansions (3.7) with (2.1), one observes a close similarity between the Lax operators and the Darboux coordinates $\xi^{[\pm]}$: they are both given by Laurent series which contain one singular term. This similarity is further strengthened by the comparison of the coefficients of these singular terms. Using (3.8) and (2.10), one finds

$$\mathcal{R} = e^{T/2} \leftrightarrow e^{\partial_s^2 F_0/2} = r, \quad (3.9)$$

in agreement with (3.4). Correspondingly, the phase space variable ω should be identified with the \mathbb{CP}^1 coordinate, $\omega^{-1} \leftrightarrow t$. Besides, the symplectic structures derived from (2.6) and $d\mathcal{X}$ (3.6) suggest that the Orlov–Shulman operators should be related to the Darboux coordinates $\tilde{\xi}^{[\pm]}$. Finally, it is well known (see, for instance, [15, 22]) that the string equations (2.12) can be considered as gluing conditions between the two patches around the north and south poles of \mathbb{CP}^1 . The condition (2.13) in turn requires that the gluing conditions generate a symplectomorphism preserving the symplectic structure. We summarize all the identifications in the following table:

ω^{-1}	\longleftrightarrow	t	
lattice variable s	\longleftrightarrow	$\rho = e^\phi$	
$\partial_s^2 F_0$	\longleftrightarrow	$T = 2 \log \mathcal{R}$	
$t_{\pm 1}$	\longleftrightarrow	z, \bar{z}	
L, \bar{L}	\longleftrightarrow	$\xi^{[\pm]}$	
$M/L, \bar{M}/\bar{L}$	\longleftrightarrow	$\tilde{\xi}^{[\pm]}$	
string equations	\longleftrightarrow	gluing conditions	(3.10)

These identifications are however a bit incomplete because we need also to match the subleading terms in the expansions of the Lax operators and Darboux coordinates. This depends on the precise matching of string equations with the gluing conditions and should be considered case by case. In particular, in the next section we consider the situation of our special interest where all identifications can be made very precise and explicit.

In addition, they say nothing about the role on the Toda hierarchy side of the third Darboux coordinate α . The point is that, in the case with an isometry, this is a derived quantity, namely, it can be given in terms of the transition functions and the explicit expressions for ξ and $\tilde{\xi}$. Therefore, it is not essential for defining the system. Nevertheless, we will see in section 5 that it also has a counterpart in the Lax formalism being related to the quasiclassical asymptotics of the Baker-Akhiezer function.

4. c-map vs $c = 1$ string

4.1 Universal hypermultiplet at tree level

Let us now consider a very special QK space obtained by the local c-map construction [8, 9], which maps a special Kähler manifold characterized by a homogeneous holomorphic prepotential $F(X)$ to a QK manifold. Physically, the local c-map produces the tree level metric on the hypermultiplet moduli space which describes the low energy effective action of type IIA string theory compactified on a Calabi-Yau manifold \mathfrak{Y} . Here we are interested in the case where the corresponding quaternionic manifold is four-dimensional. In this case the local c-map gives the moduli space of one hypermultiplet known under the name of universal hypermultiplet, which appears in the case where \mathfrak{Y} is a rigid Calabi-Yau, i.e. with vanishing Hodge number $h_{2,1}(\mathfrak{Y})$.

Since in our case the prepotential, we start with, is a holomorphic function of one variable and homogeneous of degree 2, it is always quadratic, $F(X) = \frac{\tau}{4} X^2$. Although the modulus τ carries an important physical information [33], for simplicity we restrict to the case $\tau = -\frac{i}{2}$. Then the resulting quaternionic metric is given by (3.2) where the potential T should be chosen to be one of the most trivial solutions of the Toda equation [34, 35]

$$T = \log \rho. \quad (4.1)$$

The coordinates $(\rho = e^\phi, z, \theta)$ describe respectively the dilaton, a pair of RR-fields, and the NS-axion dual in four dimensions to the antisymmetric B field.

On the other hand, the twistor description of this space is based on the covering by two patches only: \mathcal{U}_+ covers the north hemisphere of \mathbb{CP}^1 and \mathcal{U}_- covers the south one. The single transition function is determined by the prepotential,³

$$H^{[+-]} = -4 \operatorname{Im} F(\xi) = \xi^2. \quad (4.2)$$

In this case the Darboux coordinates on the twistor space have been found in [28] and, being expressed in terms of coordinates appearing in the Toda ansatz (3.2), read as follows

$$\begin{aligned} \xi^{[\pm]} &= -(z + \bar{z}) + \sqrt{\rho} (t^{-1} - t), \\ \tilde{\xi}^{[+]} &= 2(z + \sqrt{\rho} t), \quad \tilde{\xi}^{[-]} = 2(-\bar{z} + \sqrt{\rho} t^{-1}), \\ \alpha^{[\pm]} &= 2i\theta \mp (\rho - 2(\operatorname{Re} z)^2 \mp 4\sqrt{\rho}(\operatorname{Re} z)t^{\pm 1} - \rho t^{\pm 2}). \end{aligned} \quad (4.3)$$

³We adapted the normalizations to have simple duality relations.

4.2 Duality

After all these preparations, it is now trivial to establish the correspondence between the four-dimensional c-map and the perturbed $c = 1$ string at the self-dual radius whose solution is encoded in (2.21). Indeed, since the Fermi level μ is identified with the coordinate ρ , both systems are described by the same solution of the Toda equation

$$\partial_\mu^2 F_0 = T = \log \rho. \quad (4.4)$$

Moreover, it is trivial to check that the identifications (3.10) imply

$$\begin{aligned} x_+ = L &= \xi^{[+]} + \frac{1}{2} \tilde{\xi}^{[+]}, & x_- = \bar{L} &= -\xi^{[-]} + \frac{1}{2} \tilde{\xi}^{[-]}, \\ M/L &= \frac{1}{2} \tilde{\xi}^{[+]}, & \bar{M}/\bar{L} &= \frac{1}{2} \tilde{\xi}^{[-]}. \end{aligned} \quad (4.5)$$

As it should be, the string equations (2.20) appear then as a consequence of the gluing condition

$$\tilde{\xi}^{[+]} - \tilde{\xi}^{[-]} = -2\xi^{[\pm]}. \quad (4.6)$$

These equations provide the completion of the identifications in the general dictionary (3.10) and establish the full equivalence of the two systems.

In fact, the relations (4.5) are not very illuminating. One can obtain much nicer identifications if one introduces an additional patch on the twistor space of the c-map. Let it be an intermediate patch \mathcal{U}_0 around the equator of \mathbb{CP}^1 (see Fig. 1) and the associated transition functions be [28]

$$H^{[+0]} = 2iF(\xi), \quad H^{[-0]} = 2i\bar{F}(\xi). \quad (4.7)$$

This patch becomes especially important when one incorporates instanton corrections to the hypermultiplet moduli space [36] and has an advantage that the Darboux coordinates in \mathcal{U}_0 transform nicely under symplectic rotations [37]. They are given explicitly by

$$\begin{aligned} \xi &\equiv \xi^{[0]} = -(z + \bar{z}) + \sqrt{\rho} (t^{-1} - t), \\ \tilde{\xi} &\equiv \tilde{\xi}^{[0]} = (z - \bar{z}) + \sqrt{\rho} (t^{-1} + t), \\ \tilde{\alpha} &\equiv 2i \left(2\alpha^{[0]} + \xi^{[0]} \tilde{\xi}^{[0]} \right) = \sigma + 4i\sqrt{\rho} (zt^{-1} + \bar{z}t), \end{aligned} \quad (4.8)$$

where we denoted $\sigma = -8\theta - 2i(z^2 - \bar{z}^2)$. Applying (4.5), one finds remarkably simple relations between the Darboux coordinates $(\xi, \tilde{\xi})$ and the coordinates on the MQM phase space (x, p)

$$\xi = x_+ - x_- = \sqrt{2}p, \quad \tilde{\xi} = x_+ + x_- = \sqrt{2}x. \quad (4.9)$$

5. NS5-branes and the Baker-Akhiezer function

In the previous section we clarified the meaning of the Darboux coordinates ξ and $\tilde{\xi}$ under the correspondence with the $c = 1$ string. Does the third coordinate α have some interpretation on the dual side? To answer this question, let us combine (3.6) and (3.5) and integrate the resulting relation over a contour on \mathbb{CP}^1 with a free upper limit. This leads to

$$\alpha = 2\rho \log t + \int \xi d\tilde{\xi} + \text{const}, \quad (5.1)$$

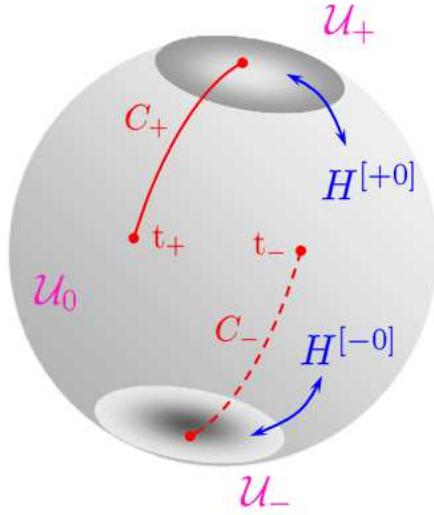


Figure 1: Covering of \mathbb{CP}^1 by three patches, the associated transition functions, and the two contours incorporating the effects of NS5-branes.

where the last term is independent of t . Then the dictionary (3.10) implies that the second term should be related to the phase of the quasiclassical Baker-Akhiezer function (2.17).

This expectation indeed turns out to be correct and can be made quite precise in our particular case characterized by the relations (4.9). Since the string equations imply that $M/L = x_-$ and $\bar{M}/\bar{L} = x_+$, the phases of the two Baker-Akhiezer functions, Ψ and $\bar{\Psi}$, are given by⁴

$$S(x_+) = \int x_- dx_+, \quad \bar{S}(x_-) = - \int x_+ dx_-. \quad (5.2)$$

Then using (2.21) and (4.8), it is easy to verify that

$$\begin{aligned} -\frac{i}{8}\sigma + S(x_+) &= \frac{1}{8} \left(-i\tilde{\alpha} + \tilde{\xi}^2 - \xi^2 \right) - \rho \log t, \\ -\frac{i}{8}\sigma + \bar{S}(x_-) &= -\frac{1}{8} \left(i\tilde{\alpha} + \tilde{\xi}^2 - \xi^2 \right) - \rho \log t. \end{aligned} \quad (5.3)$$

This confirms the identification of α (or a combination thereof with other coordinates) with the WKB phase and shows that the coordinate σ on the hypermultiplet moduli space can be seen as a common constant imaginary phase in the Baker-Akhiezer functions.

Furthermore, the result (5.3) has a remarkable consequence. It is based on the observation that the terms in the brackets on the r.h.s. appear in the exponential of the holomorphic functions

$$H_{k,\pm}^{\text{NS5}} \sim e^{\mp \pi i k \tilde{\alpha} + \pi k (\tilde{\xi}^2 - \xi^2)}, \quad (5.4)$$

which have been shown to incorporate NS5-brane instanton corrections to the classical geometry of the universal hypermultiplet [27]. Indeed, let C_{\pm} be the two contours connecting $t = 0$ ($t = \infty$) to the point t_{\pm} , corresponding to the complex submanifold $\xi \mp \tilde{\xi} = 0$ of the twistor space and found to be

$$t_+ = -\frac{z}{\sqrt{\rho}}, \quad t_- = \frac{\sqrt{\rho}}{\bar{z}}. \quad (5.5)$$

⁴The constant term can be evaluated using (2.18).

We depicted these contours in Fig. 1. Then, being integrated over C_{\pm} , the holomorphic functions $H_{k,\pm}^{\text{NS5}}$ produce exponential corrections of the following form

$$\int_{C_{\pm}} \frac{dt}{t} H_{k,\pm}^{\text{NS5}} \sim e^{-4\pi k(\rho+z\bar{z}) \mp \pi i k \sigma}. \quad (5.6)$$

This is a typical behavior of NS5-brane instantons since in our conventions $\rho \sim g_s^{-2}$ and σ is the axionic coupling associated with NS5-branes [38]. The parameter k is an integer interpreted as NS5-brane charge.

The holomorphic functions (5.4) are supposed to be considered as transition functions modifying the complex contact structure of the twistor space of the c-map.⁵ Our results suggest that, up to a universal factor $t^{\rho/\hbar}$, they can be identified with (the quasiclassical limit of) the Backer-Akhiezer functions, provided the Planck constant and the NS5-brane charge are related as

$$\hbar^{-1} = 8\pi k. \quad (5.7)$$

Amazingly, a similar relation between a quantization parameter and the NS5-brane charge has been suggested in [19] on the basis of a beautiful connection between the geometric quantization of cluster varieties [39], motivic wall-crossing formula [40, 41], and complex contact structure.

Let us recall also that in MQM the two Backer-Akhiezer functions can be seen as perturbed one-fermion wave functions in the two chiral representations where x_+ and x_- appear as diagonal operators, respectively. Thus, the wave function in the right representation encodes the effects due to NS5-branes of charge k and the (conjugate) wave function in the left representation does the same for anti-branes. The precise relations read as follows

$$H_{k,+}^{\text{NS5}} \sim t^{\rho/(i\hbar_{\text{MQM}})} \Psi(x_+) e^{-\pi i k \sigma}, \quad H_{k,-}^{\text{NS5}} \sim (t^{\rho/(i\hbar_{\text{MQM}})} \bar{\Psi}(x_-))^* e^{\pi i k \sigma}, \quad (5.8)$$

where the complex conjugation does not act on the fields and at the end one should take $\hbar_{\text{MQM}} = (8\pi i k)^{-1}$ (see footnote 1). This identification is in the beautiful agreement with a general expectation that the NS5-brane partition function should behave as a wave function [42, 43, 44, 45] and, furthermore, suggests that NS5-branes might have a fermionic nature.

Moreover, the duality with MQM provides an interesting interpretation for the contours C_{\pm} . In [27] they have been chosen by hand to produce the correct instanton effects (5.6) and their meaning was completely unclear. On the other hand, in terms of the MQM chiral coordinates, these contours turn out to be very simple: they join $x_{\pm} = \infty$ to $x_{\mp} = 0$. The starting points correspond to the asymptotics of the Fermi sea, whereas the end points are nothing else but the intersection points of the Fermi level with the lines of the perturbative stability. Namely, the inverse oscillator potential $V = -\frac{1}{2}x^2$ has two infinite “wells”. Ignoring non-perturbative effects, the movement of a particle is confined to one of them provided its initial conditions satisfy $x^2 - p^2 = 2x_+x_- > 0$. Thus, the contours C_{\pm} correspond to the parts of the Fermi profile which bounds the fermions leaking to the other side of the potential. We demonstrated this situation on Fig. 2.

⁵A new feature here comparing to what we presented above is that these transition functions are not associated to an overlap of any two open patches. In that case they would be integrated over a closed contour surrounding one of the two open patches. Instead, the contours C_{\pm} are open. Such contours typically arise when the Darboux coordinates develop branch cut singularities. In particular, this is the case for the twistor space description of D-brane instanton corrections to the local c-map [36] and BPS instanton corrections to the rigid c-map [3].

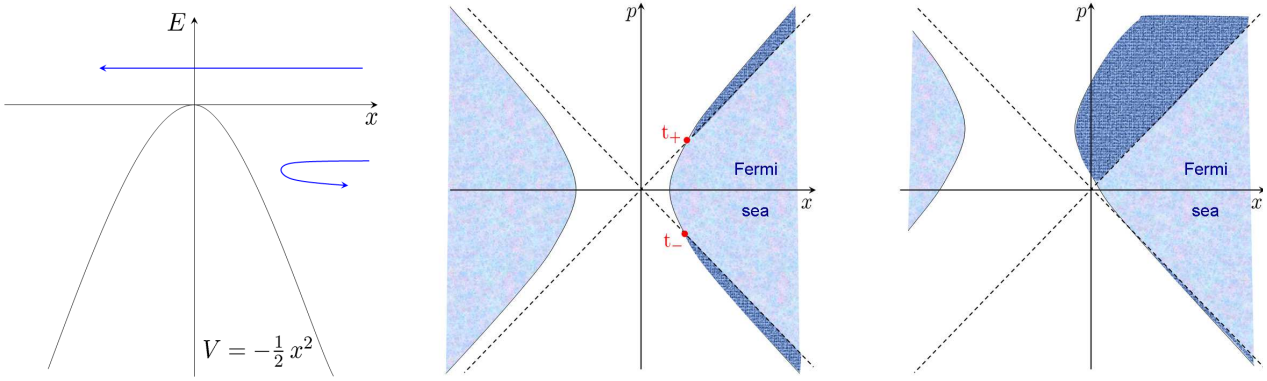


Figure 2: The movement of free fermions in the inverse oscillator potential. The second picture shows the profile of the Fermi sea on the two sides of the potential for the choice of the couplings $t_{\pm 1} = \mp 1$. t_{\pm} are the points of intersection with the lines $x_{\mp} = 0$. They bound the parts of the Fermi sea (marked by a darker color), with the phase space coordinates satisfying $x_+ x_- < 0$, which leave to (come from) the opposite side of the potential. This is illustrated on the third picture which shows the time evolution of the Fermi sea from the second picture. One can see how a part of the sea leaks from the right to the left.

5.1 Inclusion of the one-loop correction

It is very easy to include in the above description the one-loop g_s -correction to the universal hypermultiplet geometry [46, 47]. As was shown in [28], in the twistor formalism it appears as a non-vanishing anomalous dimension c , which is fixed by the Euler characteristic of the Calabi-Yau, $c = -\frac{\chi_{\mathcal{Y}}}{192\pi}$. This anomalous dimension has only two effects: it adds a logarithmic term to α (see (3.7)) and shifts the contact potential by a constant. As discussed below (3.8), this amounts to shifting in all above expressions ρ by the constant c . In particular, the one-loop corrected metric corresponds to the solution of the Toda equation $T = \log(\rho + c)$ and therefore leads to a modification of our dictionary by setting $\mu = \rho + c$. The only place where ρ remains unshifted is the relation (5.3) between α and the WKB phases since the shift is canceled by the logarithmic correction to α .

The relation between the NS5-brane instantons and the Baker-Akhiezer function works in a similar way as above. One should only take into account that the function (5.4) should be multiplied by $(\tilde{\xi} \mp \xi)^{8\pi c k}$ [27]. As a result, the relation (5.8) generalizes to

$$H_{k,+}^{\text{NS5}} \sim (t^\rho x_-^c)^{8\pi k} \Psi(x_+) e^{-\pi i k \sigma}. \quad (5.9)$$

6. Discussion

In this note we proposed a correspondence between the QK space given by the local c-map and the matrix model description of the $c = 1$ string theory with the sine-Liouville perturbation compactified at the self-dual radius. This duality is based on a dictionary between the twistorial description of four-dimensional quaternionic spaces having an isometry and the Lax formulation of the dispersionless Toda hierarchy. Under this duality, the coordinates parametrizing the quaternionic space appear as couplings of the $c = 1$ string theory, whereas

the metric is encoded in the τ -function of the hierarchy. Besides, the coordinate parametrizing the \mathbb{CP}^1 fiber of the twistor space appears on the matrix model side as a uniformization parameter of the Riemann surface given by a complexification of the profile of the Fermi sea build from the free fermions of the matrix model.

While this curious duality is very interesting, it would become useful if it can be applied to extract some information about physical effects on one side from the known results on the other side. In particular, it is very important to understand whether it can be extended to include quantum effects. While the inclusion of the one-loop correction to the c-map is almost trivial, we also provided an identification of the one-instanton corrections due to NS5-branes with the Baker-Akhiezer function of the Toda hierarchy, which is realized in the $c = 1$ string theory as the wave function of a free fermion in the chiral representation. This relation points towards a fermionic interpretation of NS5-branes which might have important implications for the study of their dynamics. In particular, one might expect that multi-instanton effects should be expressible in terms of a Slater determinant.

On the other hand, it is not clear how D-brane instantons to the hypermultiplet geometry can fit into this correspondence (see [32] for a review of the current understanding of the non-perturbative effects in the hypermultiplet sector). The problem is that there are two string couplings in the game which have different scaling. One is the coupling constant of the $c = 1$ string theory which scales as $g_s^{c=1} \sim \mu^{-1}$. The second string coupling describes quantum corrections to the hypermultiplet moduli space and is set by the dilaton to $g_s \sim \rho^{-1/2}$. Recalling that $\mu = \rho$, one obtains that, if D-instanton corrections to the c-map have a counterpart, it should scale as $e^{-1/\sqrt{g_s^{c=1}}}$, which seems to be rather strange.

Moreover, one of the key ingredients determining the structure of instanton corrections on the hypermultiplet moduli space is the Heisenberg symmetry. In particular, it requires that the coordinate z , describing the RR-fields, lives on a torus. Our correspondence then implies that the sine-Liouville couplings $t_{\pm 1}$ should be periodic. It is not clear at all how such periodicity condition can arise in the $c = 1$ string theory.

Thus, if the duality can be extended to include quantum effects, it should involve a non-trivial deformation of the $c = 1$ string and may even require to replace it by a different physical system. Some indications towards this conclusion can be seen already here. First of all, note that the NS5-brane instantons (5.6) have the same strength as the tunneling effects in the free fermionic system given by MQM. Due to this, one may expect that the fermions should fill both “wells” of the potential, which leads to consideration of the supersymmetric 0B two-dimensional string theory [48]. Second, there is a problem that the $c = 1$ string and the c-map impose different reality conditions on the Toda times: $t_{\pm 1}^* = t_{\pm 1}$ and $t_1^* = t_{-1}$, respectively. This hints that the proper system dual to the c-map is not the $c = 1$ string, but the so-called Normal Matrix Model. It is given by the same solution of the Toda hierarchy, but describes a different real section of the same holomorphic structure [49, 50].

It is important also to understand whether the presented results can be extended beyond the restriction to four dimensions. The main complication appearing in higher dimensions is that quaternionic geometries do not coincide anymore with self-dual Einstein spaces. Nevertheless, one may hope that the Toda hierarchy still describes them upon switching on higher times $t_{\pm k}$. Even if this is not the case, there is still a possibility that an extension of this hierarchy to a more general one will do the job.

So far we were looking for an interpretation of the physical effects in the hypermultiplet sector of type II string theory from the point of view of the $c = 1$ string or its MQM realization. One can consider also the opposite problem. In particular, in the $c = 1$ string theory with the sine-Liouville perturbation the non-perturbative corrections have been studied in [51, 52].⁶ One of the nice aspects of the resulting picture is that the instantons are associated with singularities of the Riemann surface already mentioned above [54, 55]. It would be interesting to see whether such geometric interpretation can be translated to the twistor space of the non-perturbative moduli spaces in $N = 2$ theories.

Another issue to be understood is that on the Toda or $c = 1$ string side, the presented duality involves so far only the dispersionless limit. As was reviewed in section 2.1, there is a natural quantization of this classical structure generating a perturbative expansion in \hbar . In $c = 1$ string theory the \hbar -corrections are interpreted as contributions from worldsheets of higher genera. A natural question is: Is there a meaning of a similar deformation for quaternionic geometries?

There are several indications that this deformation corresponds to the inclusion of NS5-brane instantons. Indeed, the Heisenberg symmetry mentioned above, in the presence of a non-vanishing NS5-brane charge, becomes equivalent to the quantum torus algebra [45]. Therefore, one can expect that switching on this charge corresponds to promoting certain structures from classical to quantum and this quantization is equivalent to the one in the Toda hierarchy. This expectation is strengthened by the observation that our duality implies the relation (5.7) between the NS5-brane charge and the Planck constant. Note also that the Toda hierarchy has been shown to incorporate the quantum torus algebra in several physically relevant situations [56].

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⁶It should be noted that most of the studies of $c = 1$ string theory with the sine-Liouville perturbation have been done for arbitrary compactification radius R . At the same time, the limit $R \rightarrow 1$ is singular and a special care is required to extract finite results both in the CFT and MQM approaches [53].

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